JOURNAL OF APPROXIMATION THEORY 64, 281–290 (1991)

Lebesgue Constants for Periodic Hermite Spline Interpolation Operators on Uniform Lattices

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Communicated by G. Meinardus

Received October 20, 1989; revised July 23, 1990

We derive a complex line integral representation for the Čebyshev norm of periodic spline interpolation operators of odd degree on uniform lattices. Several generalizations are indicated. © 1991 Academic Press, Inc

1

We consider the problem of Hermite interpolation for polynomial splines of degree $2k+1$ on a uniform lattice which without loss of generality is assumed to be given by the integers $\mathbb{Z}$. If the spline function s is required to have period $N \geqslant 1$ and to interpolate derivatives up to the order $r-1$, $r \geqslant 1$, i.e.,

$$s^{(\rho)}(v) = y_v^{(\rho)}, \quad v = 0(1)\, N-1,\ \rho = 0(1)\, r-1, \tag{1}$$

then this problem is known to be well-posed provided s is of continuity class $C^{(2k-r+1)}(\mathbb{R})$ and satisfies the consistency condition $r \leqslant k+1$ (cf. [6, 9]). After a suitable shifting of coordinates N successive polynomial components of s may be represented by the vector

$$\mathbf{q}(t) = (q_1(t), ..., q_N(t))^T, \qquad 0 \leqslant t \leqslant 1.$$

Particularly (cf. [6, 9]), if we let $\zeta := \exp 2\pi i/N$, the components $q_{j,\rho}(t)$, $j = 1(1)\, N$, $\rho = 0(1)\, r-1$, of the r fundamental splines

$$\mathbf{q}_\rho(t) := (q_{1,\rho}(t), ..., q_{N,\rho}(t)), \qquad \rho = 0(1)\, r-1,$$

0021-9045/91 $3.00

which are defined by the interpolation conditions

$$q_{j,\rho}^{(\sigma)}(0)=\delta_{\rho\sigma}\,\delta_{j1}; \qquad \rho,\sigma=0(1)\,r-1,\ j=1(1)N,$$

are given by

$$q_{j,\rho}(t)=\frac{1}{N}\sum_{\mu=0}^{N-1}\zeta^{-j\mu}h_{2k+1,r}^{(\rho)}(t,\zeta^{\mu}); \qquad j=1(1)N,\ \rho=0(1)\,r-1. \tag{2}$$

The functions $h_{2k+1,r}^{(\rho)}$ in (2) are linear combinations of the Euler–Frobenius polynomials $H_m(t,z)$ (cf. [6]). For each fixed r, $1\leqslant r\leqslant k+1$, they have a representation

$$h_{2k+1,r}^{(\rho)}(t,z)=\sum_{\sigma=0}^{r-1}\alpha_{2k+1,r}^{(\sigma,\rho)}(z)\,H_{2k+1-\sigma}(t,z). \tag{3}$$

The coefficients $\alpha_{2k+1,r}^{(\sigma,\rho)}(z)$ in (3) are, for each fixed $\rho=0(1)\,r-1$, defined as solutions of the linear systems

$$\frac{\partial^{\sigma}}{\partial t^{\sigma}}h_{2k+1,r}^{(\rho)}(t,z)\big|_{t=1}=\delta_{\rho\sigma}; \qquad \rho,\sigma=0(1)\,r-1. \tag{4}$$

The common determinant $\Delta_{2k+1,r}(z)$ of the linear systems (4)—which does not depend on ρ—can for $1\leqslant r\leqslant k+1$ be written as (cf. [2])

$$\Delta_{2k+1,r}(z)=(-1)^{[r/2]}\,0!\,1!...(r-1)!\,(1-z)^{(r-1)(4k-r+4)/2}\,H_{2k+1,r}(1,z). \tag{5}$$

The generalized Euler–Frobenius polynomials $H_{2k+1,r}(1,z)$ which occur in (5) are reciprocal polynomials of degree $2(k-r+1)$ in z with simple real zeros of sign $(-1)^r$ (cf. [3]). Particularly,

$$H_{2k+1,1}(1,z)=H_{2k+1}(1,z)$$

and

$$H_{2k+1,r}(1,\pm 1)\neq 0.$$

EXAMPLE. For $r=2$ we obtain

$$\begin{aligned}(1-z)^{2k}\,H_{2k+1,2}(1,z)\,h_{2k+1,2}^{(0)}(t,z)\\ =(2k+1)\,H_{2k}(1,z)\,H_{2k}(t,z)-2kH_{2k+1}(1,z)\,H_{2k+1}(t,z)\end{aligned}$$

and

$$\begin{aligned}(1-z)^{2k+1}\,H_{2k+1,2}(1,z)\,h_{2k+1,2}^{(1)}(t,z)\\ =H_{2k}(1,z)\,H_{2k+1}(t,z)-H_{2k+1}(1,z)\,H_{2k}(t,z)\end{aligned} \tag{6}$$

with

$$H_{3,2}(1, z) = 1$$
$$H_{5,2}(1, z) = 1 - 6z + z^2$$
$$H_{7,2}(1, z) = 1 - 72z + 262z^2 - 72z^3 + z^4$$
$$\dots$$

In the special case $r = k = 2$ we arrive at

$$\begin{aligned}(z^2 - 6z + 1)\, h^{(0)}_{5,2}(t, z) &= 4(z-1)(z^2+4z+1)\, t^5 - 5(z-1)(3z^2+8z+1)\, t^4 \\ &\quad + 20z(z-1)(z+1)\, t^3 - 10z(z-1)^2\, t^2 + z(z^2-6z+1)\end{aligned}$$

and

$$\begin{aligned}(z^2 - 6z + 1)\, h^{(1)}_{5,2}(t, z) &= (z^3+11z^2+11z+1)\, t^5 - (4z^3+33z^2+22z+1)\, t^4 \\ &\quad + 2z(3z^2+14z+3)\, t^3 - 4z(z-1)(z+1)\, t^2 + z(z^2-6z+1)\, t. \qquad (7)\end{aligned}$$

Analogously, for $r = 2$, $k = 3$,

$$\begin{aligned}H_{7,2}(1, z)\, h^{(0)}_{7,2}(t, z) &= 6(z-1)(z^4+26z^3+66z^2+26z+1)\, t^7 \\ &\quad - 7(z-1)(5z^4+104z^3+198z^2+52z+1)\, t^6 \\ &\quad + 84z(z-1)(z^3+14z^2+14z+1)\, t^5 \\ &\quad - 105z(z-1)^2(z^2+6z+1)\, t^4 + 70z(z-1)(z^3-7z^2-7z+1)\, t^3 \\ &\quad - 21z(z-1)^2(z^2-22z+1)\, t^2 + z(z^4-72z^3+262z^2-72z+1)\end{aligned}$$

and

$$\begin{aligned}H_{7,2}(1, z)\, h^{(1)}_{7,2}(t, z) &= (z^5+57z^4+302z^3+302z^2+57z+1)\, t^7 \\ &\quad - (6z^5+285z^4+1208z^3+906z^2+114z+1)\, t^6 \\ &\quad + 3z(5z^4+176z^3+478z^2+176z+5)\, t^5 \\ &\quad - 20z(z-1)(z^3+20z^2+20z+1)\, t^4 \\ &\quad + 5z(3z^4-8z^3-158z^2-8z+3)\, t^3 \\ &\quad - 6z(z-1)(z^3-31z^2-31z+1)\, t^2 \\ &\quad + z(z^4-72z^3+262z^2-72z+1)\, t. \qquad (8)\end{aligned}$$

2

Let $L^N_{2k+1,r}$ denote the spline interpolation operator which assigns to each incidence matrix $\mathbf{Y} := ((y_\nu^{(\rho)}))$, $\nu = 0(1)N-1$, $\rho = 0(1)r-1$, that N-periodic spline function $s = L^N_{2k+1,r}\mathbf{Y} \in C^{(2k+1-r)}(\mathbb{R})$ of degree $2k+1$ with knots in the integers which is characterized by the interpolation properties (1).

The Čebyshev norm of $L^N_{2k+1,r}$ is defined by

$$\|L^N_{2k+1,r}\| := \sup_{\|\mathbf{Y}\| \leqslant 1} \|L_{2k+1,r}\mathbf{Y}\|_\infty;$$

here, $\|\mathbf{Y}\| := \max\{|y_\nu^{(\rho)}|, \nu = 0(1)N-1, \rho = 0(1)r-1\}$. It is easily seen (cf. [4, 10]) that

$$\|L^N_{2k+1,r}\| = \max_{0 \leqslant t \leqslant 1} \sum_{\rho=0}^{r-1} \sum_{j=1}^{N} |q_{j,\rho}(t)|. \tag{9}$$

Consequently, any evaluation of (9) must start with an investigation of the sign behaviour of the components $q_{j,\rho}(t)$ of the fundamental splines $\mathbf{q}_\rho(t)$ for $0 \leqslant t \leqslant 1$. This has already been done for $r = 2$ in the case of cardinal Hermite interpolation by Lipow [4]. His arguments may be extended to the present more general case, but we freely omit some of the more cumbersome yet elementary details of this process.

In addition, we mention that another and perhaps generally more convenient method of solving this problem is available due to the fact that virtually all the intrinsic properties of the Hermite fundamental splines $\mathbf{q}_\rho(t)$ may be obtained as (usually simple) consequences of a transformation of their component's complex representation (2) into a real one (cf. [9] and Remark (i) in Section 3 of the present paper for some details).

3

We now present a detailed discussion of the two most simple cases: $r = 2$ and $r = 3$. The extension of our procedure to the case of arbitrary r is generally obvious.

In order to avoid any concealment of the arguments used by the necessity of handling too many different cases we categorically go, with only a slight loss of generality, on the assumption that N is even, i.e., $N = 2m$.

1. *The case* $r=2$. The components of the fundamental spline $\mathbf{q}_0(t)$ have the property

$$|q_{j,0}(t)| = q_{j,0}(t), \qquad j = 1(1)N,$$

whereas for the components of $\mathbf{q}_1(t)$ we have

$$|q_{j,1}(t)| = \begin{cases} q_{j,1}(t) & \text{for } j = 1(1)m \\ -q_{j,1}(t) & \text{for } j = m+1(1)N. \end{cases}$$

As a consequence, (2) and (9) give

$$\begin{aligned} \|L^{2m}_{2k+1,2}\| &= \max_{0 \leq t \leq 1} \frac{1}{2m} \Bigg[\sum_{j=1}^{2m} \sum_{\mu=0}^{2m-1} \zeta^{-j\mu} h^{(0)}_{2k+1,2}(t, \zeta^\mu) \\ &\quad + \sum_{j=1}^{m} \sum_{\mu=0}^{2m-1} \zeta^{-j\mu} h^{(1)}_{2k+1,2}(t, \zeta^\mu) - \sum_{j=m+1}^{2m} \sum_{\mu=0}^{2m-1} \zeta^{-j\mu} h^{(1)}_{2k+1,2}(t, \zeta^\mu) \Bigg] \\ &= \frac{1}{2m} \max_{0 \leq t \leq 1} \Bigg[2m h^{(0)}_{2k+1,2}(t, 1) + 4 \sum_{\substack{\mu=1 \\ \mu \text{ odd}}}^{2m-1} \frac{h^{(1)}_{2k+1,2}(t, \zeta^\mu)}{\zeta^\mu - 1} \Bigg] \\ &= 1 + \frac{2}{m} \sum_{\substack{\mu=1 \\ \mu \text{ odd}}}^{2m-1} \frac{h^{(1)}_{2k+1,2}(\frac{1}{2}, \zeta^\mu)}{\zeta^\mu - 1}. \end{aligned} \tag{10}$$

In the last instance we have made use of the fact that $h^{(0)}_{2k+1,2}(t, 1) = 1$ and the sum in brackets assumes its maximum value for $t = \frac{1}{2}$ (cf. [4]).

By a well-known argument, which is based on an application of the calculus of residues (cf. [5]), expression (10) may be transformed into the complex line integral representation

$$\|L^{2m}_{2k+1,2}\| = 1 + \frac{1}{\pi i} \oint_{C_2 - C_1} \frac{h^{(1)}_{2k+1,2}(\frac{1}{2}, z)}{z-1} \, \frac{z^{m-1}}{z^m + 1} \, dz. \tag{11}$$

Here, C_1 and C_2 denote positively oriented circles about the origin with radii $\rho_1 = \rho$ and $\rho_2 = \rho^{-1}$, respectively, where $\rho < 1$ is fixed in such a way that C_1 encloses all the $k-1$ zeros $z_1 < z_2 < \cdots < z_{k-1}$ of $H_{2k+1,2}(1, z)$ which are located in the interior of the unit circle.

According to the symmetry relation

$$h^{(\rho)}_{2k+1,r}(t, z^{-1}) = (-1)^\rho z^{r-1} h^{(\rho)}_{2k+1,r}(1-t, z)$$

the integral

$$\oint_{C_2} \frac{h^{(1)}_{2k+1,2}(\frac{1}{2}, z)}{z-1} \, \frac{z^{m-1}}{z^m+1} \, dz$$

can easily be shown to be equal to

$$\oint_{C_1} \frac{h^{(1)}_{2k+1,2}(\frac{1}{2}, z)}{z(z-1)} \frac{dz}{z^m+1},$$

and thus (11) eventually leads to

$$\|L^{2m}_{2k+1,2}\| = 1 + \frac{1}{\pi i} \oint_{C_1} \frac{1-z^m}{1+z^m} \frac{h^{(1)}_{2k+1,2}(\frac{1}{2}, z)}{z-1} \frac{dz}{z}, \tag{12}$$

an expression, whose evaluation turns out to be of the utmost simplicity.

EXAMPLES. (1) In the (trivial) case $k = 1$ we get from (6)

$$h^{(1)}_{3,2}(t, z) = (t^3 - 2t^2 + t)z + (t^3 - t^2),$$

i.e.,

$$h^{(1)}_{3,2}(\tfrac{1}{2}, z) = \tfrac{1}{8}(z-1),$$

which, according to (12), results in

$$\|L^{2m}_{3,2}\| = 1 + \frac{1}{8\pi i} \oint_{|z|<1} \frac{1-z^m}{1+z^m} \frac{dz}{z} = \frac{5}{4}. \tag{13}$$

It goes without saying that (13) may easily be verified directly.

(2) For $k = 2$ we have from (7)

$$h^{(1)}_{5,2}\left(\frac{1}{2}, z\right) = \frac{1}{32} \frac{(z-1)(z^2-38z+1)}{z^2-6z+1}$$

and thus by (12) we immediately arrive at

$$\begin{aligned} \|L^{2m}_{5,2}\| &= 1 + \frac{1}{32\pi i} \oint_{C_1} \frac{1-z^m}{1+z^m} \frac{z^2-38z+1}{z^2-6z+1} \frac{dz}{z} \\ &= \frac{1}{16}\left[17 + 4\sqrt{2}\, \frac{1-(3-2\sqrt{2})^m}{1+(3-2\sqrt{2})^m}\right]. \end{aligned} \tag{14}$$

Evidently, (14) enables us to state some monotonicity properties for $\|L^{2m}_{5,2}\|$ which turn out to be completely analogous to those given earlier by Cheney and Schurer [1] in the case of Lagrange interpolation with cubic splines. Furthermore, in the case of cardinal spline interpolation (i.e., $m \to \infty$) we have

$$\|L_{5,2}^{\infty}\| = 1 + \frac{1}{32\pi i} \oint_{|z| = 1/2} \frac{z^2 - 38z + 1}{z^2 - 6z + 1} \frac{dz}{z} = \frac{17 + 4\sqrt{2}}{16}.$$

(3) In the same way, for $k = 3$, from (8) and (12) we deduce that

$$\|L_{7,2}^{2m}\| = 1 + \frac{1}{128\pi i} \oint_{|z| = 1/2} \frac{1 - z^m}{1 + z^m} \frac{z^4 - 544z^3 + 7206z^2 - 544z + 1}{z^4 - 72z^3 + 262z^2 - 72z + 1} \frac{dz}{z}.$$

According to the fact that the relevant zeros of $H_{7,2}(1, z)$ are $z_1 = 0.01466871...$ and $z_2 = 0.28330706...$, the further evaluation of this expression presents no difficulties.

Remark (i). In [9] we have deduced from (2) a real form of the components $q_{j,p}(t)$ of the basis splines $\mathbf{q}_p(t)$. As has already been noted earlier, this representation not only enables us to give another proof of the properties of $q_{j,p}(t)$ which are used in the course of our derivation of expressions for the norm of the operator $L_{2k+1,r}^N$, but also provides us with the means for a different method of computing $\|L_{2k+1,r}^N\|$ itself.

For $k = r = 2$ and $N = 2m$ the last mentioned possibility reads as follows: The components $q_{j,1}(t)$, $j = 1(1)2m$, of $\mathbf{q}_1(t)$ are given (cf. [9]) by

$$\begin{aligned} q_{j,1}(t) = {} & h_{5,2}^{(1)}(t, 0)\, \delta_{j,2m} - h_{5,2}^{(1)}(1 - t, 0)\, \delta_{1j} + (1 - z_1^{2m})^{-1} \\ & \times [z_1^{2m-j-1} \operatorname*{res}_{z = z_1} h_{5,2}^{(1)}(t, z) - z_1^{j-2} \operatorname*{res}_{z = z_1} h_{5,2}^{(1)}(1 - t, z)]. \end{aligned}$$

Here, $z_1 = 3 - 2\sqrt{2}$ and

$$\operatorname*{res}_{z = z_1} h_{5,2}^{(1)}(t, z) = 4(7 - 5\sqrt{2})\, t^2(t - 1)^2\, (2t + \sqrt{2} - 1),$$

i.e.,

$$\operatorname{res} h_{5,2}^{(1)}(\tfrac{1}{2}, z) = \tfrac{1}{4}(7\sqrt{2} - 10).$$

Together with

$$h_{5,2}^{(1)}(t, 0) = t^4(t - 1)$$

we thus have

$$q_{j,1}(\tfrac{1}{2}) = \tfrac{1}{32}(\delta_{j1} - \delta_{j,2m}) + \tfrac{1}{4}(7\sqrt{2} - 10)(1 - z_1^{2m})^{-1}\, (z_1^{2m-j-1} - z_1^{j-2}).$$

Consequently,

$$\max_{0\leqslant t\leqslant 1}\sum_{j=1}^{2m}|q_{j,1}(t)|=\sum_{j=1}^{m}q_{j,1}\left(\frac{1}{2}\right)-\sum_{j=m+1}^{2m}q_{j,1}\left(\frac{1}{2}\right)$$
$$=\frac{1}{16}+\frac{1}{2}(7\sqrt{2}-10)(1-z_1^{2m})^{-1}(z_1^m-1)\sum_{j=1}^{2m}z^j$$
$$=\frac{1}{16}+\frac{1}{2\sqrt{2}}\frac{1-z_1^m}{1+z_1^m},$$

from which, by noting that

$$\sum_{j=1}^{2m}|q_{j,0}(t)|=\sum_{j=1}^{2m}q_{j,0}(t)=1,$$

we finally obtain in accordance with (14)

$$\|L_{5,2}^{2m}\|=\frac{17}{16}+\frac{1}{2\sqrt{2}}\frac{1-z_1^m}{1+z_1^m}.$$

2. *The case* $r=3$. If N is specified in such a way that

$$N=2m, \qquad m \text{ even}, \tag{15}$$

the sign distribution for the components of the fundamental splines will be as follows:

$$|q_{j,0}(t)|=\begin{cases}(-1)^{j-1}q_{j,0}(t) & \text{for } j=1(1)m\\ (-1)^j q_{j,0}(t) & \text{for } j=m+1(1)2m,\end{cases} \tag{16}$$

$$|q_{j,1}(t)|=(-1)^{j-1}q_{j,1}(t) \qquad \text{for } j=1(1)2m, \tag{17}$$

$$|q_{j,2}(t)|=\begin{cases}(-1)^{j-1}q_{j,2}(t) & \text{for } j=1(1)m\\ (-1)^j q_{j,2}(t) & \text{for } j=m+1(1)2m.\end{cases} \tag{18}$$

In a similar way as in the case $r=2$, which has already been dealt with, (15) and the relations (16)–(18) successively lead to the following statements

$$\max_{0\leqslant t\leqslant 1}\sum_{j=1}^{2m}|q_{j,0}(t)|=\sum_{j=1}^{2m}\left|q_{j,0}\left(\frac{1}{2}\right)\right|$$
$$=\frac{2}{m}\sum_{\substack{\mu=1\\ \mu\text{ odd}}}^{2m-1}\frac{h_{2k+1,3}^{(0)}(\frac{1}{2},\zeta^\mu)}{1+\zeta^\mu}$$
$$=\frac{1}{\pi i}\oint_{C_1}\frac{1-z^m}{1+z^m}\frac{h_{2k+1,3}^{(0)}(\frac{1}{2},z)}{z+1}\frac{dz}{z},$$

$$\max_{0\leqslant t\leqslant 1}\sum_{j=1}^{2m}|q_{j,1}(t)|=\sum_{j=1}^{2m}(-1)^{j-1}q_{j,1}\left(\frac{1}{2}\right)$$

$$=\frac{1}{2m}\sum_{\mu=0}^{2m-1}\sum_{j=1}^{2m}(-1)^{j-1}\zeta^{-j\mu}h_{2k+1,3}^{(1)}\left(\frac{1}{2},\zeta^{\mu}\right)$$

$$=-h_{2k+1,3}^{(1)}\left(\frac{1}{2},-1\right),$$

$$\max_{0\leqslant t\leqslant 1}\sum_{j=1}^{2m}|q_{j,2}(t)|=\sum_{j=1}^{2m}\left|q_{j,2}\left(\frac{1}{2}\right)\right|$$

$$=\frac{1}{\pi i}\oint_{C_1}\frac{1-z^m}{1+z^m}\,\frac{h_{2k+1,3}^{(2)}(\frac{1}{2},z)}{z+1}\,\frac{dz}{z}.$$

C_1 now denotes a positively oriented circle with center 0 and radius $\rho<1$, where ρ is fixed in such a manner that C_1 encloses all of the $k-2$ zeros of $H_{2k+1,3}(1,z)$ with an absolute value less than or equal to one. Eventually, all these terms sum up to

$$\|L_{2k+1,3}^{2m}\|=\frac{1}{\pi i}\oint_{C_1}\frac{1-z^m}{1+z^m}$$

$$\times\frac{h_{2k+1,3}^{(0)}(\frac{1}{2},z)+h_{2k+1,3}^{(2)}(\frac{1}{2},z)}{z+1}\,\frac{dz}{z}-h_{2k+1,3}^{(1)}\left(\frac{1}{2},-1\right). \qquad (19)$$

EXAMPLES. (i) In the trivial case $k=2$

$$h_{5,3}^{(0)}(\tfrac{1}{2},z)=\tfrac{1}{2}(z+1),$$

$$h_{5,3}^{(1)}(\tfrac{1}{2},z)=\tfrac{5}{32}(z-1),$$

$$h_{5,3}^{(2)}(\tfrac{1}{2},z)=\tfrac{1}{64}(z+1)$$

and (19) results in

$$\|L_{2k+1,3}^{2m}\|=1+\tfrac{1}{32}+\tfrac{5}{16}=\tfrac{43}{32}.$$

Again, this can easily be verified by direct calculation.

(ii) For $k=r=3$ we have (cf. [8, p. 135])

$$h_{7,3}^{(0)}\left(\frac{1}{2},z\right)=\frac{1}{128}\,\frac{(z+1)(29z^2+582z+29)}{z^2+8z+1},$$

$$h_{7,3}^{(1)}\left(\frac{1}{2},z\right)=\frac{1}{128}\,\frac{(z-1)(7z^2+136z+7)}{z^2+8z+1},$$

$$h_{7,3}^{(2)}\left(\frac{1}{2},z\right)=\frac{1}{256}\,\frac{(z+1)(z^2+58z+1)}{z^2+8z+1},$$

i.e.,

$$\|L_{7,3}^{2m}\| = \frac{61}{192} + \frac{1}{256\pi i}\oint_{|z|=1/2} \frac{1-z^m}{1+z^m}\,\frac{59z^2+1222z+59}{z^2+8z+1}\,\frac{dz}{z}$$
$$= \frac{1}{384}\left[299 + 75\sqrt{15}\,\frac{1-(4-\sqrt{15})^m}{1+(4-\sqrt{15})^m}\right].$$

Remark (ii). Similar methods turn out to be useful in obtaining expressions for Lebesgue constants even in the case of more general periodic spline interpolation operators (cf. [7]).

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